

Gliders, Ether and Triangles

MARKUS REDEKER

*International Centre of Unconventional Computing, University of the West of
England, Bristol, United Kingdom.[†]*

This is a study of the about structures in one-dimensional cellular automata, with the elementary cellular automaton Rule 54 as example. It uses the formalism of “flexible time” to derive expressions that characterise triangles, gliders, and and periodic background patterns like the ether.

A theorem is derived about the existence of simple triangles in one-dimensional cellular automata and their dependence on properties of the transition rule of the automaton. For Rule 54, expressions for its triangles, three of its gliders and its ether are found and used to look into their behaviour in detail.

Key words: Rule 54, one-dimensional cellular automata, gliders, ether, triangles, flexible time

1 INTRODUCTION

The formalism of “flexible time” was invented in order to find and describe structures in the behaviour of cellular automata. This paper contains results that are motivated by the behaviour of a specific one-dimensional cellular automaton, Rule 54.

Well-known structures in the behaviour of one-dimensional cellular automata like Rule 54 are the *ether*, a periodic background pattern, and *gliders*, which move in the ether with constant speed (Figure 1). But the basic building block for both of them are the triangular structures that are created by

[†] email: markus2.redeker@live.uwe.ac.uk

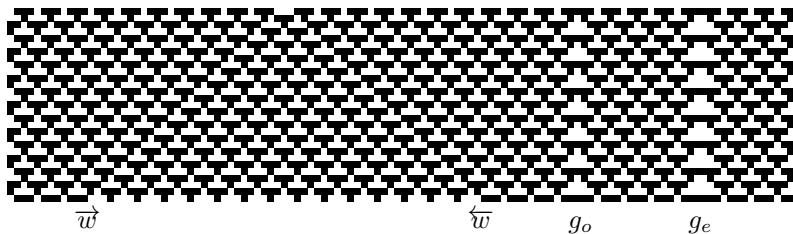


Figure 1

Periodic patterns under Rule 54. The diagram shows three types of gliders on an ether background. Time goes upward. (Since Rule 54 is symmetric, \overleftarrow{w} and \overrightarrow{w} are viewed as variants of the same particle.)

a sequence of cells in the same state (Figure 2). Triangles are a common structure of many one-dimensional cellular automata.

This paper has therefore two main parts. Section 2 concentrates on triangles and one-dimensional cellular automata in general. The formalism of flexible time is clarified and developed further, based on [2], and a sufficient condition for the existence of triangles is proved.

Section 3 is more exploratory in nature. The results of Section 2 are applied to Rule 54 and, using it as an example, expressions for periodic background patterns and for gliders in the formalism of flexible time are found, both in the specific case of Rule 54 and for cellular automata in general.

2 REACTION SYSTEMS AND TRIANGLES

2.1 Cells

The theory of one-dimensional cellular automata is about the behaviour of cells that are arranged in a line. Their position on that line is an integer, and since time is discrete, time is an integer too, and each cell has a finite number of possible states.

So before cellular automata can be formally defined, we must define cells.

Definition 1 (Cell). *Let Σ be a finite set. A cell is a pair $(p, \sigma) \in \mathbb{Z}^2 \times \Sigma$. Its position is p and its state is σ .*

In this context p is a space-time point and is understood as a pair (t, x) , where t is the time and x the space coordinate.

In this article, a cell is an object that exists only for one moment, at a specific site and at a specific time step. At the next time step a new cell is created. Therefore that what we call here a cell will in other texts be viewed as a “cell-at-a-time”. I have chosen not to follow this convention because the notion of a cell-at-a-time is more important and will be used here more often than that of a cell-over-many-timesteps.

In a cellular automaton, the behaviour of the cells follows a global rule. To describe its effect we take a step backwards and consider systems which the cells do not need to follow a law. Such systems are called here cellular processes.

Definition 2 (Cellular Process). *A cellular process is a function $\pi: D \rightarrow \Sigma$ with $D \subseteq \mathbb{Z}^2$. The set D is the domain of π .*

The proposition “ $\pi(t, x) = \sigma$ ” is interpreted as “The cell at position x and time x is in state σ ”.

Finite processes will be especially important here, much more than those with domain \mathbb{Z}^2 .

2.2 Cellular automata

The behaviour of a cellular automaton is given by its transition rule, which maps the states of the neighbourhood of a cell (consisting of the cells r positions to the left and to the right of it, and the cell itself) to the state of the cell one time step later.

Definition 3 (Transition rule). *Let Σ be a finite set and $r \geq 0$ an integer. A transition rule of a cellular automaton with radius r and state set Σ is a function*

$$\varphi: \Sigma^{2r+1} \rightarrow \Sigma. \quad (1)$$

In this article I assume that φ is not a constant function.

Next we define which cellular processes describe the behaviour of cells that follow the transition rule φ .

Let $p \in \mathbb{Z}^2$. The set $\{p + (0, i): -r \leq i \leq r\}$ contains its neighbouring points, and $p + (1, 0)$ is the point one time step later. If the cells in a cellular process π follow the transition rule φ then the state of cell at $p + (1, 0)$ is determined by φ and the states of the cells in the neighbourhood of p .

This neighbourhood may be partially outside the domain of π and contain cells whose states are unknown. Nevertheless the value of φ applied to this neighbourhood may be known – if the value of φ is independent of the values of the unknown cells.

Therefore, in the following definition, we use the set $N_\pi(p)$ of all possible neighbourhoods of a point p , given the information in π .

Definition 4 (Cellular Evolution). *Let $\pi: D \rightarrow \Sigma$ be a cellular process and φ be a transition rule for Σ with radius r . For every $p \in \mathbb{Z}^2$ define*

$$N_\pi(p) = \{ (\nu_{-r}, \dots, \nu_r) \in \Sigma^{2r+1} : \\ \text{if } p + (0, i) \in D \text{ then } \nu_i = \pi(p + (0, i)) \} . \quad (2)$$

The cellular process π is consistent with φ if

$$\pi(p) = \varphi(w) \quad \text{for all } p \in D \text{ and } w \in N_\pi(p - (1, 0)) . \quad (3)$$

It is closed under φ if also $p \in D$ for all p for which $\{ \varphi(w) : w \in N_\pi(p - (1, 0)) \}$ has exactly one element.

If there is a closed process that contains π as a subset, then the smallest such process is $\hat{\pi}$, the closure of π .

We view a cellular automaton as a kind of machine and the cellular processes as partial views of its possible behaviours. The following definition expresses this.

Definition 5 (Cellular automaton). *The cellular automaton $\text{CA}(\varphi)$ for a transition rule φ is the set of all cellular processes that are consistent with φ .*

This definition has the advantage that $\text{CA}(\varphi)$ does not change if the radius of φ is artificially extended to a larger value $r' > r$: If we define a new transition rule by $\varphi'(\sigma_{-r'}, \dots, \sigma_{r'}) = \varphi(\sigma_{-r}, \dots, \sigma_r)$, then $\text{CA}(\varphi) = \text{CA}(\varphi')$.

2.3 Cell sequences

As an intermezzo we introduce now some definitions about spatially extended sequences of cells that we will use later. All except the last are nothing new.

Definition 6 (Cell sequences). *Let Σ be a set of cell states.*

Let $s \in \Sigma^\ell$. We call s a cell sequence of length ℓ .

We write the length of s as $|s|$.

The set of all cell sequences is $\Sigma^ = \bigcup_{\ell \geq 0} \Sigma^\ell$.*

We identify cell sequences with functions $s: \{0, \dots, |s| - 1\} \rightarrow \Sigma$. Then $s(0)$ is the first component of s , $s(1)$ the second, and so on.

We may write s as a product of elements of Σ : $s = s(0)s(1) \dots s(|s| - 1)$.

Let $0 \leq i \leq i + k \leq |s|$. The subsequence of s with length k starting at i is

$$s_{i:k} = s(i)s(i+1) \dots s(i+k-1) . \quad (4)$$

2.4 Situations

The formalism of *Flexible Time* [2] is motivated by the idea that it is easier to find patterns in the evolution of cellular automata if one considers structures that involve cells at different times. These structures are called situations.

Definition 7 (Situations). *A cellular process $s: \{p + (0, i): 0 \leq i < \ell\} \rightarrow \Sigma$, with $p \in \mathbb{Z}^2$ and $\ell \geq 0$ is a segment of ℓ cells starting from p . Segments are compatible if their union is a function.*

The segment s has a size $\delta(s) = p + (0, \ell)$, i. e. $\delta(s)$ is the point right of the last cell in s . The time component of $\delta(s)$ is $\delta(s)_t$, the space component is $\delta(s)_x$.

A situation is a finite sequence of compatible segments. Situations are compatible if their segments are compatible. To $s = (s_0, \dots, s_n)$ belongs a cellular process

$$\pi_s = \bigcup_{i \in I} s_i. \quad (5)$$

The size of s is the size of its last element: $\delta(s) = \delta(s_n)$.

It is possible and useful if segments overlap. Segments with length $\ell = 0$ are also possible.

For situations we will use a notation that expresses the distances between the segments instead of their absolute positions.

Definition 8 (Notations for situations). *Let s be a segment of ℓ cells starting from p with $s(p + (0, i)) = \sigma_i$. It is written*

$$[p]\sigma_0 \dots \sigma_{\ell-1}. \quad (6)$$

A situation $s = (s_0, s_1, \dots, s_n)$ with $s_i = [p_i]w_i$ is written

$$[p_0]w_0[p_1 - \delta(s_0)]w_1, \dots, [p_n - \delta(s_{n-1})]w_n. \quad (7)$$

The empty situation is λ . For $[(t, x)]$ we write $[t, x]$, for $[0, x]$, $[x]$. A term $[0]$ at the beginning of a situation is generally omitted.

For cellular automata with radius r we use the abbreviations

$$\ominus = [-1, -r], \quad \oplus = [1, -r]. \quad (8)$$

The notation (7) is a special case of the product of situations.

Definition 9 (Product of situations). *Let $s = ([p_0]v_0, \dots, [p_m]v_m)$ and $t = ([q_0]w_0, \dots, [q_n]w_n)$ be two situations. Their product is*

$$st = ([p_0]v_0, \dots, [p_m]v_m, [q_0 + \delta(s)]w_0, \dots, [q_n + \delta(s)]w_n). \quad (9)$$

The product exists if $([q_0 + \delta(s)]w_0, \dots [q_n + \delta(s)]w_m)$ is compatible to s . With the product we get the usual notations related to it, like exponentiation, Kleene star, and so on.

2.5 Reactions

The evolution of a cellular automaton is described by *reactions*. A reaction is a pair of two situation a and b , written as $a \rightarrow b$, in which b represents a “later” state of the automaton than a . We will describe the behaviour of a cellular automaton by a small set of *generator reactions* between a small number of cells and from the *reaction system* generated by them.

Definition 10 (Reaction system). *Let φ be a transition rule.*

A reaction for φ is a pair (a, b) of situation for φ where $\hat{\pi}_a$ exists, $\hat{\pi}_a \supseteq \pi_b$ and $\delta(a) = \delta(b)$.

If (a, b) and (xay, xby) are reactions, then we say that (xay, xby) is the result of the application of (a, b) on xay .

A reaction system for φ is a pair (Φ, \rightarrow_Φ) , where Φ is a set of situations for φ and the binary relation $\rightarrow_\Phi \subseteq \Phi^2$ is a set of reactions for φ that is transitive, includes the identity and is closed under application, i. e.

if $a \rightarrow_\Phi b$ and (xay, xby) is a reaction with $xay \in \Phi$, then $xay \rightarrow_\Phi xby$ and $xby \in \Phi$.

If the context is unambiguous we write \rightarrow_Φ as \rightarrow and speak simply of “the reaction system Φ ”.

The condition $\delta(a) = \delta(b)$ ensures that in the application $xay \rightarrow_\Phi xby$ of $a \rightarrow b$ the factor y designates the same cells in xay and xby .

Definition 11 (Generators). *Let φ be a transition rule, S_0 and S be two sets of situations for φ with $S_0 \subseteq S$, and G a set of reactions for φ .*

The reaction system generated by S and G , $\langle S_0, S; G \rangle$, is the smallest reaction system (Φ, \rightarrow_Φ) for φ with $S_0 \subseteq \Phi \subseteq S$, which is closed under application of reactions from G and also closed under factoring, i. e.

$$\text{if } xay \in \Phi \text{ then } a \in \Phi. \quad (10)$$

We can work with $(\Phi, \rightarrow_\Phi) = \langle S_0, S; G \rangle$ in a purely algebraic way, without needing to check whether $\hat{\pi}_a$ exists for every $a \in \Phi$, because of the following lemma.

Lemma 12. *If $\hat{\pi}_a$ exists and (a, b) is a reaction for φ , then $\hat{\pi}_b$ exists.*

If $\hat{\pi}_{xay}$ exists, then $\hat{\pi}_a$ exists.

Proof. Both propositions follow from the fact that if $\hat{\pi}$ exists and $\psi \subseteq \hat{\pi}$, then $\hat{\psi}$ exists.

In the first case we have $\hat{\pi}_a \supseteq \pi_b$ since (a, b) is a reaction for φ , and therefore $\hat{\pi}_b$ exists. In the second case $\pi_{[\delta(x)]b} \subseteq \hat{\pi}_{xay}$, and $\hat{\pi}_{[\delta(x)]a}$ is a shifted version of $\hat{\pi}_a$. \square

2.6 The slow reaction system

The following reaction system is a relatively straightforward translation of a transition rule into the language of reaction systems. The core of the system are the reactions (11f) and (11g); the rest is needed for bookkeeping.

Definition 13 (Generator reactions). *Let φ be a transition function for Σ with radius r . The set $G(\varphi)$ consists of the following reactions for all $u, v \in \Sigma^*$:*

$$uv \rightarrow u[0]v, \quad (11a)$$

$$u[0]v \rightarrow uv, \quad (11b)$$

$$u \rightarrow u \oplus \ominus u \quad \text{if } |u| = 2r, \quad (11c)$$

$$\ominus u \oplus \rightarrow [|u| - 2r] \quad \text{if } |u| \leq 2r, \quad (11d)$$

$$u[-|u|]u \rightarrow u, \quad (11e)$$

$$\ominus u \rightarrow \sigma \ominus u_{1:|u|-1} \quad \text{if } \varphi(ux) = \sigma \text{ for all } x \in \Sigma^{2r+1-|u|}, \quad (11f)$$

$$u \oplus \rightarrow u_{0:|u|-1} \oplus \sigma \quad \text{if } \varphi(xu) = \sigma \text{ for all } x \in \Sigma^{2r+1-|u|}. \quad (11g)$$

The reactions (11a) and (11b) will often be used implicitly, to split a segment of a situation in two parts so that another reaction can be applied to it, and to unite them afterwards. The visible effect of the reactions (11c) and (11d) is to create and destroy \ominus and \oplus symbols. The destruction reactions (11d) remove also cell states that cannot be used in another reaction. To simplify overlapping segments that may occur in this process we need another bookkeeping reaction, (11e). The reactions (11f) and (11g) finally introduce new cell states; others are removed from the situation to which they are applied, such that it still can be written with Σ , \oplus and \ominus .

But we first need a proof that they are actually reactions.

Theorem 14. *The elements of $G(\varphi)$ are reactions for φ .*

Proof. Let $a \rightarrow b \in G(\varphi)$. One can see that always $\delta(a) = \delta(b)$ and that π_a involves only cells from the same time. Therefore $\hat{\pi}_a$ exists and we only have to check whether $\pi_b \subseteq \hat{\pi}_a$.

In (11a), (11b), (11c) and (11e) we have $\pi_b = \pi_a$, and in (11d) we have $\pi_b = \emptyset$, so the condition is trivially true.

For (11f) note that $\pi_b \setminus \pi_a = \pi_\sigma$. So we need to show that $\pi_\sigma \subset \hat{\pi}_a$ in order to prove that $\pi_b \subseteq \hat{\pi}_a$. We have

$$N_{\pi_a}(-1, 0) = \{ \varphi(ux) : x \in \Sigma^{2r+1-|u|} \} \quad (12)$$

and $\varphi(w) = \sigma$ for all $w \in N_{\pi_a}(-1, 0)$, therefore $(0, 0)$ must belong to the domain of $\hat{\pi}_a$ and $\hat{\pi}_a(0, 0) = \sigma$, which proves that (11f) is a reaction.

Reaction (11g) is like (11f), except that left and right are exchanged and it is shifted one time step to the future – so a mirror image of the previous argument will work for it. \square

The reaction system generated by $G(\varphi)$ gets its name from the fact that its generator reactions create at most one new cell at a time. We will use it in the next sections as building block for a faster system.

Definition 15 (Slow reaction system). *Let φ be a transition rule for Σ . The system $(\Phi, \rightarrow) = \langle \Sigma^*, (\Sigma \cup \{\ominus, \oplus\})^*; G(\varphi) \rangle$ is called the slow reaction system for φ .*

Both in terms of the size of the set of its situations and the set of its reactions this system lies between the systems Φ and Ψ defined in [2].

2.7 Triangles

We now search for reactions of larger structures. At the present stage in the development of the theory we concentrate on structures generated from periodic sequences of cells. If x is a situation then the situations x^k behave in a regular way, at least if k is large enough. The following definition captures this phenomenon for cellular processes.

Definition 16 (Triangle). *Let π be a reaction system. Let x be a situation for φ and $k_0 > 0$ such that for every $k \geq k_0$ the process $\hat{\pi}_{x^k}$ is finite. Then the set $\{ \hat{\pi}_{x^k} : k \geq k_0 \}$ is a type of triangle processes; its elements are triangle processes.*

If $\hat{\pi}_{x^k}$ is finite, it has a triangular shape, as the number of cells whose states are determined from the information in x^k becomes smaller with time: Figure 2 shows an example for Rule 54. It also shows a *triangle reaction* which traces the boundaries of the triangle process: the initial situation of the reaction is the base of the triangle, and the end situation contains its other two sides. It is actually a product of two terms, one for the left side, written with elements of Σ and \oplus signs, and the other one for the right side and written with elements of Σ and \ominus signs.

This is a general phenomenon, and we capture it with the following notation.

Definition 17 (Slopes). *Let S be a set of situations. An element $s \in S$ is a positive slope if for all factorisations $s = s_0 s_1 s_2$ we have $\delta(s_1)_t \geq 0$. It is a negative slope if always $\delta(s_1)_t \leq 0$. The subset of all positive slopes in S is S_+ , that of all negative slopes is S_- . Their intersection is S_0*

We use the convention that elements of S_- and S_+ have $+$ or $-$ as indices, so that a typical element of S_- has a name like s_- .

With it we define triangle reactions in a general context.

Definition 18 (Triangle reactions). *Let Φ be a reaction system. Let $k_0 \geq 0$, $m > 0$ be integers and $x \in \Phi$, a_+ , $y_+ \in \Phi_+$, $b \in \Phi$ and a_- , $y_- \in \Phi_-$ be situations. A set of reactions*

$$T = \{ x^{km+k_0} \rightarrow_{\Phi} a_+ y_+^k b y_-^k a_- : k \geq 0 \} \quad (13)$$

is called a type triangle reactions for x if none of the reactions in T can be applied to b . The elements of T are called triangle reactions.

x^{km+k_0} is the base of the triangle, y_-^k and y_+^k are its sides and a_- , a_+ and b are its corners. b is also called the end of the triangle. x is a base term of the triangle and y_- and y_+ are its side terms.

In the case of $a_+ = y_+$ and $a_- = y_-$, (13) can be written as

$$T = \{ x^{km+k_0} \rightarrow_{\Phi} y_+^k b y_-^k : k \geq 1 \}. \quad (14)$$

In it, y_+ and y_- are both corner and side terms.

2.8 Existence of triangles

We will consider here only the simplest kind of triangles, those where x is a single cell. Then we can pose also restrictions on the other parameters of the triangle; this will allow to deduce the existence of triangles from properties of φ .

In this section φ is always a transition rule for Σ with radius r and Φ the slow reaction system for φ of Definition 15.

Definition 19 (Simple triangles). *A triangle process $\hat{\pi}_{x^k}$ is simple if $x \in \Sigma$.*

A triangle reaction $x^{km+k_0} \rightarrow a_+ y_+^k b y_-^k a_-$ is simple if $k_0 \geq 2r$, $x \in \Sigma$, $b \in \Sigma^k$ with $k \geq 1$ and

$$a_+, x_+ \in \{ s \oplus : s \in \Phi_+ \}, \quad a_-, x_- \in \{ \ominus s : s \in \Phi_- \}. \quad (15)$$

If a triangle reaction is simple then in its result $a_+ y_+^k b y_-^k a_-$ the “latest” cells are completely contained in b . We require that k_0 is at least $2r$ because then the cells on both sides of the triangle cannot influence each other except at its end. The triangles in Figure 2 are simple.

We start the existence proof for triangles with a definition and two lemmas that connects the existence of certain reactions with properties of φ .

Definition 20 (Contraction Parameters). *Let $\sigma \in \Sigma$ and $\varphi(\sigma^{2r+1}) = \tau$.*

Let $c_L(\sigma)$ and $c_R(\sigma)$ be the largest integers such that

$$\begin{aligned} \varphi(w\sigma^{c_L(\sigma)+r+1}) &= \tau \quad \forall w \in \Sigma^{r-c_L(\sigma)}, \\ \varphi(\sigma^{c_R(\sigma)+r+1}w) &= \tau \quad \forall w \in \Sigma^{r-c_R(\sigma)}. \end{aligned} \quad (16)$$

The contraction parameters of σ are $c_L(\sigma)$, $c_R(\sigma)$ and $c(\sigma) = c_L(\sigma) + c_R(\sigma)$.

Lemma 21 (Values of the contraction parameters). *Let $\sigma \in \Sigma$. Then*

$$|c_L(\sigma)| \leq r, \quad |c_R(\sigma)| \leq r, \quad |c(\sigma)| \leq 2r. \quad (17)$$

Proof. From (16) we see that $-r-1 \leq c_L(\sigma) \leq r$. But the case $c_L(\sigma) = -r-1$ can only occur if $\varphi(w) = \tau$ for all w . But then φ is constant, which was excluded. The same reasoning is valid for c_R . \square

Lemma 22 (Contraction parameters and reactions). *Let $\tau = \varphi(\sigma^{2r+1})$. Then Φ has the reactions*

$$\begin{aligned} \sigma^{c_L(\sigma)+r+1} \oplus &\rightarrow \sigma^{c_L(\sigma)+r} \oplus \tau, \\ \ominus \sigma^{c_R(\sigma)+r+1} &\rightarrow \tau \ominus \sigma^{c_R(\sigma)+r}. \end{aligned} \quad (18)$$

Proof. The reactions in (18) are derived from the generator reactions (11f) and (11g). The only open question is whether the exponents of σ are all nonnegative. (Otherwise the situations involved do not exist.) But this follows from the restrictions of the contraction parameters in Lemma 21. \square

Next we look for reactions that reach over only one time step, but can be indefinitely extended spatially.

Definition 23 (Layers of a triangle). *Let $\sigma, \tau \in \Sigma$. A set of reactions*

$$\{ \sigma^k \rightarrow \sigma^{c_L(\sigma)+r} \oplus \tau^{k-c(\sigma)} \ominus \sigma^{c_R(\sigma)+r} : k \geq k_0 \} \quad (19)$$

is a (σ, τ) -layer with minimal width k_0 .

Note that $\delta(\sigma^{c_L(\sigma)+r} \oplus)_x = c_L(\sigma)$ and $\delta(\ominus \sigma^{c_R(\sigma)+r})_x = c_R(\sigma)$; with this equation the validity of (19) can be easily be checked. It is from (19) that the contraction parameters got their names: they show how much shorter the τ -sequence of a (σ, τ) -layer is than the σ -sequence.

Lemma 24. *For each $\sigma \in \Sigma$ exists a $(\sigma, \varphi(\sigma^{2r+1}))$ -layer with minimal width of at least $2r$.*

Proof. Let $\tau = \varphi(\sigma^{2r+1})$. With Lemma 22 we have

$$\begin{aligned} \sigma^{2r} &\rightarrow \sigma^{2r} \oplus \ominus \sigma^{2r} \\ &\rightarrow \sigma^{c_L(\sigma)+r} \oplus \tau^{r-c_L(\sigma)} \tau^{r-c_R(\sigma)} \ominus \sigma^{c_R(\sigma)+r} \\ &= \sigma^{c_L(\sigma)+r} \oplus \tau^{2r-c(\sigma)} \ominus \sigma^{c_R(\sigma)+r} \end{aligned} \quad (20)$$

Let now $k \geq 2r$. Then we have

$$\begin{aligned} \sigma^k &\rightarrow \sigma^{c_L(\sigma)+r} \oplus \tau^{2r-c(\sigma)} \ominus \sigma^{c_R(\sigma)+r+(k-2r)} \\ &\rightarrow \sigma^{c_L(\sigma)+r} \oplus \tau^{k-c(\sigma)} \ominus \sigma^{c_R(\sigma)+r}. \end{aligned} \quad (21)$$

□

The following theorem covers the simplest case of triangles in one-dimensional cellular automata, especially the triangles in Rule 54.

Theorem 25 (Existence of simple triangles). *Let $(\sigma_i)_{i \geq 0}$ be a sequence of elements of Σ such that $\varphi(\sigma_i^{2r+1}) = \sigma_{i+1}$ and $c(\sigma_i) \geq 0$ for all $i \geq 0$, and such that $\sigma_i = \sigma_{i+p}$ for $i \geq i_0$. If*

$$\sum_{i=0}^{p-1} c(\sigma_{i_0+p}) > 0 \quad (22)$$

then Φ has a simple triangle whose base consists of σ_0 .

Proof. For every i exists a (σ_i, σ_{i+1}) -layer with the reactions

$$\sigma_i^n \rightarrow \sigma_i^{c_L(\sigma_i)+r} \oplus \sigma_{i+1}^{n-c(\sigma_i)} \ominus \sigma_i^{c_R(\sigma_i)+r} \quad (23)$$

If n is large enough, then there is another layer reaction that starts with $\sigma_{i+1}^{n-c(\sigma_i)}$, to which another reaction of type (23) may be applied, and so on, until the resulting cell sequence is too short. The resulting reaction, starting at σ_j^n and running over ℓ steps, is

$$\sigma_i^n \rightarrow s_+(i, \ell) \sigma_{i+\ell}^{n-c(i, \ell)} s_-(i, \ell). \quad (24)$$

Here we use the notations

$$s_+(i, \ell) = \prod_{j=i}^{i+\ell-1} (\sigma_j^{\text{CL}(\sigma_j)+r} \oplus), \quad (25)$$

$$s_-(i, \ell) = \prod_{j=i+\ell-1}^i (\ominus \sigma_j^{\text{CR}(\sigma_j)+r}) \quad (26)$$

for the sides, and

$$c(i, \ell) = \sum_{j=i}^{i+\ell-1} c(\sigma_j) \quad (27)$$

for the length of the resulting cell sequence.

Reaction (24) is already something like a triangle; we must show that it becomes ultimately periodical and that there is for every i a value of ℓ such that to the no layer reaction can be applied to $\sigma_{i+\ell}^{n-c(i,\ell)}$.

Since the sequence of the σ_i is ultimately periodic with period p , there are p different kinds of triangles that start from σ_0 , depending on the state of the cells at the top layer. Therefore we now define terms for the triangle sides and corners that are indexed by $i \in \{0, \dots, p-1\}$ such that e. g. a_{i+} belongs to the triangle ending in state c_{i_0+i} :

$$a_{i+} = s_+(0, i_0 + i), \quad a_{i-} = s_-(0, i_0 + i), \quad (28)$$

$$x_{i+} = s_+(i_0 + i, p), \quad x_{i-} = s_-(i_0 + i, p) \quad (29)$$

The reactions for them are

$$\sigma_0^n \rightarrow a_{i+} \sigma_{i_0+i}^{n-c(0, i_0+i)} a_{i-}, \quad (30)$$

$$\sigma_{i_0+i}^n \rightarrow x_{i+} \sigma_{i_0+i+p}^{n-c(i_0+i, p)} x_{i-} = x_{i+} \sigma_{i_0+i}^{n-c(i_0, p)} x_{i-}. \quad (31)$$

For (31) we have used that $(\sigma_i)_{i \geq 0}$ is ultimately periodic with period p : This means that $\sigma_{i_0+i+p} = \sigma_{i_0+i}$ and $c(i_0 + i, p) = c(i_0, p)$.

Since $c(i_0, p) > 0$ – this is condition (22) – we can write $n = kc(i_0, p) + k_0$ with $k \geq 0$ and $0 \leq k_0 < c(i_0, p)$. Therefore reaction (31) can be iterated at most k times. The resulting reaction is

$$\sigma_{i_0+i}^{kc(i_0, p)+k_0} \rightarrow x_{i+}^k \sigma_{i_0+i}^{k_0} x_{i-}^k. \quad (32)$$

We now set $m = c(i_0, p)$ to stay in the terminology of Definition 18. Then there are two possibilities: If $i_0 = i = 0$ we have the triangle reaction type

$$\{ \sigma_0^{km+k_0} \rightarrow x_{0+}^k \sigma_0^{k_0} x_{0-}^k : k \geq 1 \} \quad \text{for } 1 \leq k_0 \leq m \quad (33)$$

Otherwise we use (30) and get

$$\{ \sigma_0^{km+k_1} \rightarrow a_{i+} x_{i+}^k \sigma_{i_0+i}^{k_0} x_{i-}^k a_{i-} : k \geq 0 \} \quad \text{for } 0 \leq k_0 \leq m \quad (34)$$

with $k_1 = k_0 + c(0, i_0 + i)$. \square

2.9 Triangle systems

In this section the triangle reactions, which were found in the slow reaction system Φ , are supplemented by other reactions to create a new system, Θ , which is based on triangles and in which all reactions consists of families which involve an arbitrarily large number of cells.

The triangle reactions create “valleys” between the triangles, structures of the following form:

Definition 26 (Valley). *Let S be a set of situations and $s \in S$. If there is a decomposition*

$$s = s_0 y_- f y_+ s_1 \quad (35)$$

with $s_0, s_1 \in S$, $y_- \in S_- \setminus S_0$, $f \in S_0$ and $y_+ \in S_+ \setminus S_0$, then we say that s has the valley $y_- f y_+$. The number $\delta(s_0 y_-)_t$ is the level of the valley.

The following lemma shows that valleys are quite common. This means that it should be enough to extend triangle reactions with reactions that start from valleys to get a working reaction system.

Lemma 27. *Let S be a set of situations and $s \in S \setminus S_+ S_-$. Then s has a valley.*

Proof. If $s \notin S_+ S_-$ it must have one or more factorisations $s = s_0 y_- f' y'_+ s'_1$ with $s_0, f', s'_1 \in S$, $y_- \in S_- \setminus S_0$, $y'_+ \in S_+ \setminus S_0$. Choose among these factorisations one for which $\delta(s_0 y_-)_t$ is minimal. (It exists because s is finite.) Then f' must begin with an element of S_+ . So we can find a factorisation $f' = f y_+ g$ with $f \in S_0$, $y_+ \in S_+ \setminus S_0$ and $g \in S_+$. Now we set $s_1 = g y'_+ s'_1$; the result is (35). \square

The following definition shows such a system. It contains additional conditions to the reactions to make sure that two kinds of reactions are actually enough.

Definition 28 (Reaction system with triangles). *Let X and Y be two disjoint sets of situations with $X = X_0,^* \lambda \notin Y$, and let $Z \subseteq Y$ be a set of situations with $\delta(z_-)_t > 0$ for every $z_- \in Z_-$.*

* This cryptic equation is a way to express that $\Theta_0 = X^*$ in the resulting reaction system. We will also have $\Theta_- = Y_-^*$ and $\Theta_+ = Y_+^*$.

1. Connection identities. Assume that for all $a_- \in Y_-$ and $a_+ \in Y_+$ there is one equation

$$a_- = a'_- z_- \quad \text{or} \quad a_+ = z_+ a'_+ \quad (36)$$

with $a'_- \in Y_- \cup \{\lambda\}$, $a'_+ \in Y_+ \cup \{\lambda\}$, $z_- \in Z_-$ and $z_+ \in Z_+$.

2. Assume that there are the following two types of generator reactions:

- (a) Triangle reactions. For every $x \in X$ a family of reactions

$$x^{km+k_x} \rightarrow a_+ y_+^k b y_-^k a_- \quad k \geq 0, \quad (37)$$

with $x \in X$, $b \in X^*$, a_+ , $y_+ \in Y_+$ and a_- , $y_- \in Y_-$.

Define the set F as those elements of X to which none of the reactions in (37) can be applied. The elements of F are called fusion terms.

- (b) Fusion reactions. For all $z_- \in Z_-$, $f \in F$ and $z_+ \in Z_+$ one reaction

$$z_- f z_+ \rightarrow g \quad (38)$$

with $g \in X^*$.

These reactions form the set G_Θ .

3. Shielding conditions. Assume that the following conditions are valid, both as written and with left and right reversed:

- (a) If there is a fusion reaction $z_- f z_+ \rightarrow gx$, and the triangle reaction for base x has corner z_+ and side term y_+ , then $z_+ \neq y_+$.
(b) If there is a fusion reaction $z_- f z_+ \rightarrow gx$ and an $a_+ \in Y_+$ so that $z_+ a_+ \in Y_+$, then there is no triangle reaction with base x and corner a_+ .

4. Define the set R_Θ as the subset of $(X \cup Y)^*$ which does not contain any term xa_+ or a_-x from any triangle reaction (37) as factor.

The reaction system $(\Theta, \rightarrow_\Theta) = \langle X^*, R_\Theta; G_\Theta \rangle$ is called a simple triangle system.

As a result of the restrictions in R_Θ the triangle reactions must always be applied fully. A reaction like $x^{km+k_x} \rightarrow a_+ y_+^{k-1} b y_-^{k-1} a_- x^m$ would result in a situation that is not an element of Θ . The shielding conditions ensure that no

term xa_+ or a_-x can occur as the result of a reaction. The advantage of such restrictions is that we then need no reactions that extend triangles later.

Simple triangle systems have a complete set of reactions, in terms of the following theorem:

Theorem 29 (Completeness of triangle systems). *Let $(\Theta, \rightarrow_\Theta)$ be a simple triangle system and $s \in \Theta$.*

Then there is either a reaction $s \rightarrow_\Theta x_+x_-$ with $x_+ \in \Theta_+$ and $x_- \in \Theta_-$ or there is for every $t \geq 0$ a reaction $s \rightarrow_\Theta s'$ such that all valleys in s' are at a level greater than t .

Proof. In this proof we extend the notion of “level” to all elements of Θ : if there is a decomposition $s = s_0ys_1$, then the level of y in s is $\delta(s_0y_-)_t$,

First assume that $v = y_-fy_+$ is a valley in s at level t . If we apply a triangle reaction to f , then the elements of Θ_- it can create are at a level of at least t . If f is a fusion sequence and we apply a fusion reaction to v , then the resulting term must be at a higher level than t .

Let now s be any element of $\Theta \setminus \Theta_+\Theta_+$. It must have valleys. Let the minimal level of the valleys in s be t_0 . If we find all valleys at level t_0 in s and first apply all possible triangle reactions to them and then fusion reactions to the resulting valleys, then all elements of Θ_- created this way must be at a higher level than t_0 . All valleys in the reaction product must have been there before or contain an Θ_- term created in the reaction. If there are valleys in the reaction result, they must therefore occur at a higher level than t_0 .

So for every $s \in \Theta \setminus \Theta_+\Theta_+$ there is a reaction $s \rightarrow_\Theta s'$ such that either s' contains valleys and their level is strictly higher than the minimal level of the valleys in s , or that $s' \in \Theta_+\Theta_-$. In the first case we can find a reaction of the same type starting from s' . If we repeat this, we either arrive at situations of arbitrarily high minimal level, or stop at an element of $\Theta_+\Theta_-$. \square

3 RULE 54

“Rule 54” is the common name – following the naming convention of Stephen Wolfram [3] – of a one-dimensional cellular automaton with two states and a three-cell neighbourhood.

3.1 Basic properties

Rule 54 has $\Sigma = \{0, 1\}$, $r = 1$ and

$$\varphi(s) = \begin{cases} 1 & \text{for } s \in \{001, 100, 010, 101\}, \\ 0 & \text{otherwise.} \end{cases} \quad (39)$$

	$\ominus 000 \rightarrow 0\ominus 00$	$000\oplus \rightarrow 00\oplus 0$
	$\ominus 001 \rightarrow 1\ominus 01$	$100\oplus \rightarrow 10\oplus 1$
	$\ominus 010 \rightarrow 1\ominus 10$	$010\oplus \rightarrow 01\oplus 1$
	$\ominus 011 \rightarrow 00\ominus 1$	$110\oplus \rightarrow 1\oplus 00$
	$\ominus 10 \rightarrow 1\ominus 0$	$01\oplus \rightarrow 0\oplus 1$
	$\ominus 11 \rightarrow 0\ominus 1$	$11\oplus \rightarrow 1\oplus 0$
	$00 \rightarrow 00\oplus\ominus 00$	$\ominus 00\oplus \rightarrow \lambda$
	$01 \rightarrow 01\oplus\ominus 01$	$\ominus 1\oplus \rightarrow [-1]$
	$10 \rightarrow 10\oplus\ominus 10$	
	$11 \rightarrow 11\oplus\ominus 11$	
Shortcuts	$\ominus 010 \rightarrow 11\ominus 0$	$010\oplus \rightarrow 0\oplus 11$
	$01 \rightarrow 0\oplus 1\ominus 01$	
	$10 \rightarrow 10\oplus 1\ominus 0$	
	$11 \rightarrow 1\oplus 11\ominus 1$	

Table 1
Generator reactions for the slow reaction system for Rule 54.

A simple way to memorise this rule is to express it as

“ $\varphi(s) = 1$ if s contains at least one 1, except if the ones touch each other (as in 011).”

Note that φ is symmetric under the interchange of left and right.

3.2 Slow reaction system

Table 1 shows the most important generator reactions from $G(\varphi)$ for Rule 54. Not all reactions from (11f) and (11g) are shown, only those with minimal value of k . Thus we have $\ominus 11 \rightarrow 0\ominus 1$ (which is derived from $\varphi(110) = \varphi(111) = 0$) but not $\ominus 110 \rightarrow 0\ominus 10$. The shorter reactions can now be applied on the results of some others: $\ominus 010 \rightarrow 1\ominus 10$, a reaction of G_4 , is then extended by $\ominus 10 \rightarrow 1\ominus 0$ to $\ominus 010 \rightarrow 11\ominus 0$. There is a list of such reactions at the bottom of the table.

3.3 Triangle reaction system

Next we derive the triangles and the simple triangle reaction system Θ for Rule 54.

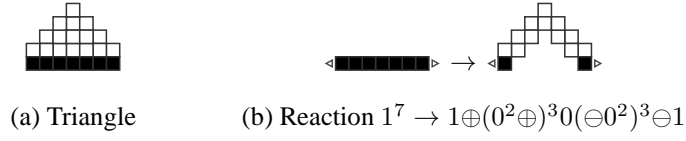


Figure 2
Triangles and triangle reactions.

Rule 54, as a symmetrical rule, has $c_L = c_R$. For it, the values of the contraction parameters and the resulting reactions are

$$\begin{aligned} c_L(0) = c_R(0) = 1, \quad \ominus 000 \rightarrow 0 \ominus 00, \quad 000 \oplus \rightarrow 00 \oplus 0, \\ c_L(1) = c_R(1) = 0, \quad \ominus 11 \rightarrow 0 \ominus 1, \quad 11 \oplus \rightarrow 1 \oplus 0. \end{aligned} \quad (40)$$

Therefore we have a $(0, 0)$ -layer and a $(1, 0)$ -layer, namely

$$\begin{aligned} \{ 0^k \rightarrow 0^2 \oplus 0^{k-2} \ominus 0^2 : k \geq 3 \}, \\ \{ 1^k \rightarrow 1 \oplus 0^k \ominus 1 : k \geq 2 \}. \end{aligned} \quad (41)$$

They cause the following triangle reaction types,

$$\begin{aligned} \{ 0^{2k+k_0} \rightarrow (0^2 \oplus)^k 0^{k_0} (\ominus 0^2)^k : k \geq 1 \} & \quad \text{for } k_0 \in \{1, 2\}, \\ \{ 1^{2k+k_1} \rightarrow 1 \oplus (0^2 \oplus)^k 0^{k_1} (\ominus 0^2)^k \ominus 1 : k \geq 0 \} & \quad \text{for } k_1 \in \{2, 3\}. \end{aligned} \quad (42)$$

Since the side terms of the triangles will occur often, we use abbreviations for them, namely

$$\epsilon_- = \ominus 0^2, \quad \epsilon_+ = 0^2 \oplus, \quad a_- = \ominus 1, \quad a_+ = 1 \oplus. \quad (43)$$

Figure 3 shows diagrams for them and for other situations that have a special name.

With the abbreviations (42) becomes

$$\begin{aligned} \{ 0^{2k+k_0} \rightarrow \epsilon_+^k 0^{k_0} \epsilon_-^k : k \geq 1 \} & \quad \text{for } k_0 \in \{1, 2\}, \\ \{ 1^{2k+k_1} \rightarrow a_+ \epsilon_+^k 0^{k_1} \epsilon_-^k a_- : k \geq 0 \} & \quad \text{for } k_1 \in \{2, 3\}. \end{aligned} \quad (44)$$

Therefore we have $X = \{0, 1\}$, $Y = \{\epsilon_+, a_+, \epsilon_-, a_-\}$ and $Z = Y$. The set R_Θ contains all the situations that do not contain $\epsilon_- 0$, $0 \epsilon_+$, $a_- 1$ or $1 a_+$ as subsequences.

$$\begin{array}{lll}
\epsilon_- = \ominus 00: & \begin{array}{c} \blacktriangleleft \square \triangleright \end{array} & w_- = \epsilon_- 1: \begin{array}{c} \blacktriangleleft \square \blacksquare \triangleright \end{array} & e_- = \epsilon_- \ominus 1: \begin{array}{c} \blacktriangleleft \square \blacksquare \triangleright \end{array} \\
\epsilon_+ = 00 \oplus: & \begin{array}{c} \blacktriangleleft \square \triangleright \end{array} & w_+ = 1 \epsilon_+: \begin{array}{c} \blacktriangleleft \blacksquare \square \triangleright \end{array} & e_+ = 1 \oplus \epsilon_+: \begin{array}{c} \blacktriangleleft \blacksquare \triangleright \end{array} \\
g_o = e_+ \epsilon_+ 0 \epsilon_- e_-: & \begin{array}{c} \blacksquare \square \square \blacksquare \\ \square \square \square \square \\ \square \square \square \square \end{array} & g_e = e_+ \epsilon_+ 0^2 \epsilon_- e_-: & \begin{array}{c} \blacksquare \square \square \blacksquare \\ \square \square \square \square \\ \square \square \square \square \end{array}
\end{array}$$

Figure 3

Useful situations in Rule 54. Next to the algebraic expression for a situation s is a diagram for π_s , together with markers \blacktriangleleft for the point $(0, -1)$ and \blacktriangleright for $\delta(s)$. They show the points where the other factors end or begin if s is factor in a larger product. Cells in the states 0 and 1 appear as \square and \blacksquare .

The set F of all fusion sequences for contains all elements of Σ^* that do not have the minimal triangle bases 0^3 or 1^2 as factor. Then if $f \in F$,

$$\ominus f \oplus \rightarrow 1^{|f|-2}. \quad (45)$$

One can see that by noticing that every 3-cell neighbourhood in an end-sequence for Rule 54 must be 001, 010, 100 or 101; and for all of them the value of the transition rule φ is 1.

With this we get

$$\epsilon_- f \epsilon_+ \rightarrow 1^{|f|+2} \quad |f| \geq 1, \quad 0^2 f 0^2 \in F, \quad (46a)$$

$$\epsilon_- f a_+ \rightarrow 1^{|f|+1} \quad |f| \geq 0, \quad 0^2 f 1 \in F, \quad (46b)$$

$$a_- f \epsilon_+ \rightarrow 1^{|f|+1} \quad |f| \geq 0, \quad 1 f 0^2 \in F, \quad (46c)$$

$$a_- f a_+ \rightarrow 1^{|f|} \quad |f| \geq 1, \quad 1 f 1 \in F. \quad (46d)$$

These are all possible fusion reactions.

We must check whether the shielding conditions are true for them. For the first condition we notice that all fusion reactions are of the form $z_- f z_+ \rightarrow 1^k$, so we need to check whether the triangle reaction with base term 1 has

States:	$0, 1, \epsilon_-, \epsilon_+, a_-, a_+$.
Situations:	No subsequences $\epsilon_-0, 0\epsilon_+, a_-1, 1a_+$.
Triangles:	$0^{2k+k_0} \rightarrow \epsilon_+^k 0^{k_0} \epsilon_-^k \quad k \geq 1, k_0 \in \{1, 2\}$ $1^{2k+k_1} \rightarrow a_+ \epsilon_+^k 0^{k_1} \epsilon_-^k a_- \quad k \geq 0, k_1 \in \{2, 3\}$
Fusion reactions:	$\epsilon_- f \epsilon_+ \rightarrow 1^{ f +2} \quad 0^2 f 0^2 \in F, \quad f \geq 1$ $\epsilon_- f a_+ \rightarrow 1^{ f +1} \quad 0^2 f 1 \in F, \quad f \geq 0$ $a_- f \epsilon_+ \rightarrow 1^{ f +1} \quad 1 f 0^2 \in F, \quad f \geq 0$ $a_- f a_+ \rightarrow 1^{ f } \quad 1 f 1 \in F, \quad f \geq 1$

Table 2

The simple triangle reaction system for Rule 54. F is the set of fusion sequences, as defined in the text.

different corner and side terms. It has the corner a_+ and the side term ϵ_+ , so the condition is fulfilled for the left side. Since the rule is symmetric, it is also true for the right side. The second condition is true because no product of an element of Z and an element of Y is an element of Y .

Therefore the resulting reaction system is a simple triangle system. It is listed in Table 2.

3.4 Ether and gliders

Now we will use the new reaction system to look at some phenomena that occur under Rule 54. The first of them is the *ether*, a robust background pattern. It consists at alternating time steps of either the cell sequence 01^3 or 10^3 infinitely repeated. (To verify the reactions in this section Table 3 may be helpful.)

A formula for the ether can be derived from the 01^3 generation: We have

$$01^3 \rightarrow 0a_+\epsilon_+0\epsilon_-a_-, \quad (47)$$

so we introduce the names

$$e_- = \epsilon_-a_-, \quad e_+ = \epsilon_+a_+. \quad (48)$$

Then we get (see Figure 4)

$$0e_-0e_+ \rightarrow 0e_+0e_-, \quad (49)$$

$\epsilon_- e_+ \rightarrow 1\epsilon_+$	$\epsilon_- e_+ \rightarrow w_+$
$e_- \epsilon_+ \rightarrow \epsilon_- 1$	$e_- \epsilon_+ \rightarrow w_-$
$\epsilon_- 1\epsilon_+ \rightarrow e_+ 0e_-$	$w_- e_+ \rightarrow e_+ 0e_-$
	$e_- w_+ \rightarrow e_+ 0e_-$
$e_- 0e_+ \rightarrow e_+ 0e_-$	
$\epsilon_- 10e_+ \rightarrow 1^3\epsilon_+$	$w_- 0e_+ \rightarrow 1^3\epsilon_+$
$e_- 10\epsilon_+ \rightarrow 1^3\epsilon_+$	$e_- 0w_+ \rightarrow 1^3\epsilon_+$
$\epsilon_- 101\epsilon_+ \rightarrow 1^5$	$w_- 0w_+ \rightarrow 1^5$
$\epsilon_- 10^2 1\epsilon_+ \rightarrow 1^6$	$w_- 0^2 w_+ \rightarrow 1^6$

Table 3

Reactions that are useful in Section 3.4. Most of them are special cases of Table 2 or derived from them.

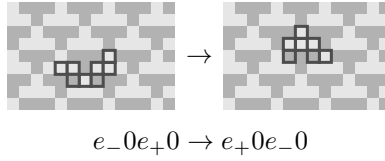


Figure 4

How the ether reaction fits into the development if the ether. The cells that belong to the reaction are marked.

which leads together with (47) to

$$(01^3)^k \rightarrow (0e_+)^k (0e_-)^k \quad k \geq 0, \quad (50)$$

a very simple triangle reaction. This is in contrast to the other possible starting point, 10^3 , where one gets

$$(10^3)^k \rightarrow 1\epsilon_+ (0e_+)^k (0e_-)^k \epsilon_- 1 \quad k \geq 1, \quad (51)$$

a more complicated triangle reaction, in which also the components of the other ether phase, e_- and e_+ , reappear. The reaction system selects thus one of the phases of the ether as more natural than the other, which is a helpful simplification.

There is a reaction for ϵ_- and ϵ_+ that is similar to (49),

$$\epsilon_- \epsilon_+ \rightarrow \epsilon_+ \epsilon_- . \quad (52)$$

Both are *background reactions* of the form

$$b_- b_+ \rightarrow b_+ b_- . \quad (53)$$

This reaction can easily be iterated to $b_-^k b_+^\ell \rightarrow b_+^\ell b_-^k$, which describes the evolution of a large piece of a periodic background pattern.

There are three kinds of long-lived structures that are described in [1] in some detail. There they are called particles, now usually *gliders*. There is one moving particle w , which appears as \overleftarrow{w} and \overrightarrow{w} , depending on the direction in which it moves, and the “odd” and “even gutter” g_o and g_e , which are immobile.

The w particle “may be generated by three 0’s followed by three 1’s or the converse” [1, p. 870]. We try this now and get

$$0^3 1^3 \rightarrow \epsilon_+ 0 \epsilon_- e_+ 0 e_- \rightarrow \epsilon_+ 0 \ 1 \epsilon_+ \ 0 e_- . \quad (54)$$

In it we can recognise $0 e_-$ as a part of the ether and $\epsilon_+ 0$ as a part of the ether in the wrong phase (as in (49) and (51)), so the rest must be the w particle. Therefore we define

$$w_- = \epsilon_- 1, \quad w_+ = 1 \epsilon_+ . \quad (55)$$

These definitions must be verified: We must show that w actually moves through the ether. But we have

$$w_- 0 e_+ 0 = \epsilon_- 1 0 e_+ 0 \rightarrow 1^3 \epsilon_+ 0 \rightarrow e_+ 0 e_- \epsilon_+ 0 \rightarrow e_+ 0 \epsilon_- 1 0 = e_+ 0 w_- 0, \quad (56)$$

which shows how w_- is destroyed and reappears at the right of its previous position (Figure 5). w_- is therefore stable and corresponds to the right-moving glider \overrightarrow{w} of [1].

The two immobile gliders, g_e and g_o , are in fact small triangles, as can be seen from the pictures in [1]. It turns out that the right definitions for them are

$$g_o = e_+ \epsilon_+ 0 \epsilon_- e_-, \quad g_e = e_+ \epsilon_+ 0^2 \epsilon_- e_- . \quad (57)$$

The verification that they actually behave like gliders is straightforward (Fig-

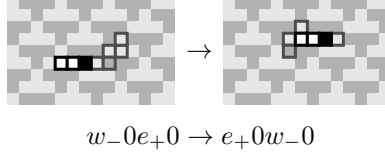


Figure 5
A \vec{w} glider moving on an ether background. The w_- part is emphasised.

ure 6),

$$\begin{aligned}
 e_-0g_o0e_+0 &= e_-0e_+\epsilon_+0\epsilon_-e_-0e_+0 \\
 &\rightarrow e_+0 e_- \epsilon_+0\epsilon_-e_+0 e_-0 \\
 &\rightarrow e_+0 w_-0w_+0 e_-0 \\
 &\rightarrow e_+0 1^50 e_-0 \\
 &\rightarrow e_+0 e_+\epsilon_+0\epsilon_-e_-0 e_-0 = e_+0g_o0e_-0, \tag{58}
 \end{aligned}$$

$$\begin{aligned}
 e_-0g_e0e_+0 &= e_-0e_+\epsilon_+0^2\epsilon_-e_-0e_+0 \\
 &\rightarrow e_+0 e_- \epsilon_+0^2\epsilon_-e_+0 e_-0 \\
 &\rightarrow e_+0 w_-0^2w_+0 e_-0 \\
 &\rightarrow e_+0 1^60 e_-0 \\
 &\rightarrow e_+0 e_+\epsilon_+0^2\epsilon_-e_-0 e_-0 = e_+0 g_e0 e_-0, \tag{59}
 \end{aligned}$$

but the appearance of the w gliders in the process is a bit surprising. It suggests the interpretation that the gliders g_o and g_e decay into two w gliders, which then collide and create its next incarnation. With flexible time the gliders suddenly have an internal structure.

The three glider reactions (56), (58) and (59) have again a common structure, which can be described by the *glider condition*

$$b_-^k g b_+^\ell \rightarrow b_+^\ell g b_-^k. \tag{60}$$

Here b_- and b_+ form a background pattern as in (53) and g is the glider. The number $(\ell - k)/(\ell + k)$ is a measure for the speed of the glider.

We have now already touched the creation of other gliders by the w gliders. Of the two syntheses found in the behaviour of the g particles, the first one,

$$w_-0w_+ \rightarrow g_o, \tag{61}$$

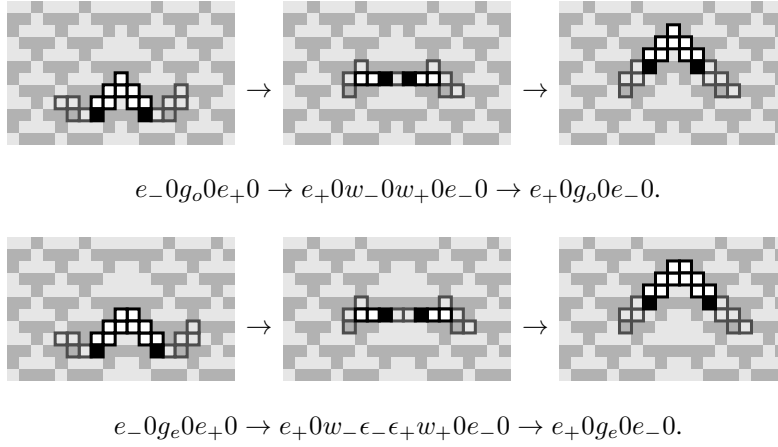


Figure 6
Evolution of the g_o and g_e gliders, together with the intermediate states where the w gliders appear.

is more important because here the w gliders are at the right distance to have been part of the ether before. Such a glider synthesis has been already noticed in [1], but here it occurs as a corollary of a previous analysis.

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